

# DIHEDRAL GROUP AND REPETITIVE ACHROMATS WITH MIRROR SYMMETRIC OR MIRROR ANTISYMMETRIC BASIC CELL

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## Abstract

Using the group-theoretical point of view we study in this paper second and third order repetitive achromats with a mirror symmetric or mirror antisymmetric basic cell and compare these achromats with repetitive achromats designed without internal cell symmetries taken into account.

## INTRODUCTION

As an achromat we understand a particle transport system whose linear transfer matrix is dispersion free (first-order achromat) and whose transfer map does not have nonlinearities of transverse motion up to a certain order  $m$  ( $m$ -order achromat). The first practical solution for the second-order achromat was presented at the end of 1970s in the paper [1], where the theory of achromats based on repetitive symmetry was developed, and quickly becomes part of many accelerator designs. The way to construct the third and higher order repetitive achromats was pointed out about ten years later in [2] and was based on normal form theory. The capabilities of the mirror (reversal) symmetry for the cancellation of nonlinear aberrations were explored in [3] and, taking additionally into account the magnifying achromat [4] and the four-cell staircase achromat mentioned, for example, in [5], the theory of achromats eventually came to the state when one has a set of different design recipes without a single theoretical framework and no answer to the question, if there is a magnet arrangement which will give better automatic cancellation of aberrations than those already known.

The way to create the unified theory of achromats was found in [6] by looking at the design of the symmetry based achromats from the point of view of the theory of finite matrix groups. This approach not only explains why all previously known designs work, but also says that there is no better cancellation of aberrations than cancellation provided by the action of finite cyclic or dihedral group (at least as long as the linear transverse oscillations are uncoupled). In this paper we compare the actions of cyclic and dihedral groups using as test bed second and third order repetitive achromats with a mirror symmetric or mirror antisymmetric basic cell. Note that these achromats are new by itself and were not considered before with the exception of the two-cell case, which was studied in [3] under simplifying assumption that the cell matrix is dispersion free.

## VARIABLES, MAPS AND ACHROMATS

We consider the beam dynamics in a magnetostatic system which is symmetric about the horizontal midplane  $y = 0$  and use a complete set of symplectic variables  $z = (x, p_x, y, p_y, \sigma, \varepsilon)^\top$  as particle coordinates. In this set the variables  $\hat{z} = (x, p_x, y, p_y)^\top$  describe the transverse

particle motion and the variables  $\sigma$  and  $\varepsilon$  characterize the longitudinal dynamics [6]. We represent particle transport from one longitudinal location to another by a symplectic map and assume that for arbitrary two longitudinal positions the point  $z = 0$  is the fixed point and that the corresponding map can be Taylor expanded in its neighborhood. We use that up to any predefined order  $m$  the aberrations of a map  $\mathcal{M}$  can be represented through a Lie factorization as

$$: \mathcal{M} : =_m \exp(: \mathcal{F}_{3,m+1} :) : M :, \quad (1)$$

where  $\mathcal{F}_{3,m+1} \stackrel{\text{def}}{=} \mathcal{F}_3 + \dots + \mathcal{F}_{m+1}$ , each of the functions  $\mathcal{F}_k$  is a homogeneous polynomial of degree  $k$  in the variables  $z$  and the symbol  $=_m$  denotes equality up to order  $m$  (inclusive) when maps on both sides of (1) are applied to the phase space vector  $z$ .

We also use that the map  $\mathcal{M}$  of a magnetic system which is symmetric about the horizontal midplane  $y = 0$  satisfies

$$: \mathcal{M} :: T_M :: T_M :: \mathcal{M} :, \quad (2)$$

where  $T_M = \text{diag}(1, 1, -1, -1, 1, 1)$  is the mid-plane symmetry matrix. From equality (2), time-independence, energy conservation and symplecticity it follows that the polynomial  $\mathcal{F}_{3,m+1}$  in (1) does not depend on the variable  $\sigma$  and is an even function of the variables  $y$  and  $p_y$ , and that the matrix  $M$  in (1) has the following form

$$M = \begin{pmatrix} r_{11} & r_{12} & 0 & 0 & 0 & r_{16} \\ r_{21} & r_{22} & 0 & 0 & 0 & r_{26} \\ 0 & 0 & r_{33} & r_{34} & 0 & 0 \\ 0 & 0 & r_{43} & r_{44} & 0 & 0 \\ r_{51} & r_{52} & 0 & 0 & 1 & r_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3)$$

Using the representation (1) we can state that the map  $\mathcal{M}$  is a  $m$ -order achromat if, and only if, the matrix of its linear part  $M$  is dispersion free (i.e.  $r_{16} = r_{26} = 0$ ) and the polynomial  $\mathcal{F}_{3,m+1}$  is a function of the variable  $\varepsilon$  only.

## DISPERSION DECOMPOSITION

Let us consider a midplane symmetric cell with the map

$$: \mathcal{M}_c : =_3 \exp(: \mathcal{F}_{3,4}^c(z) :) : M_c :, \quad (4)$$

and let us assume that the cell transfer matrix  $M_c$  allows the solution for the periodic (matched) dispersion to be found. If we denote by  $A$  and  $B$  the initial conditions for the periodic cell dispersion and its derivative respectively, then the cell transfer matrix can be represented in the form

$$M_c = D_c N_c D_c^{-1}, \quad (5)$$

where the matrix  $N_c$  is dispersion-free and its  $4 \times 4$  upper left block  $\hat{M}_c$  coincides with the corresponding block of the matrix  $M_c$ , and the matrix  $D_c$  can be expressed in the form of a Lie operator as follows

$$: D_c : = \exp(: \varepsilon (B x - A p_x) :). \quad (6)$$

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Using the dispersion decomposition (5) the cell transfer map (4) can be brought into the form

$$: \mathcal{M}_c : =_3 : D_c :^{-1} \exp(: \mathcal{P}_{3,4}^c(\mathbf{z}) :) : N_c : : D_c : \quad (7)$$

with  $\mathcal{P}_{3,4}^c(\mathbf{z}) = \mathcal{F}_{3,4}^c(x + A \cdot \varepsilon, p_x + B \cdot \varepsilon, y, p_y, \varepsilon)$ .

Let us now turn our attention to the special features of the dispersion decompositions of cells with symmetries.

The matrix of a system which is mirror symmetric about the  $x - y$  plane to the original system is given by

$$M_R = T_R M^{-1} T_R, \quad (8)$$

where  $T_R = \text{diag}(1, -1, 1, -1, -1, 1)$  is the reversion transformation matrix, and the matrix of the system which is mirror antisymmetric about the  $x - y$  plane to the original system (reversed and then rotated by  $180^\circ$  around the longitudinal axis) is given by

$$M_A = T_A M^{-1} T_A, \quad (9)$$

where  $T_A = \text{diag}(-1, 1, -1, 1, -1, 1)$ .

Let us consider a mirror symmetric cell where the system with the transfer matrix  $M$  is followed by its reversed image and let us assume that the  $r_{21}$  element of the matrix  $M$  is not equal to zero. Then the cell transfer matrix  $M_c = M_R M$  allows dispersion decomposition (5) with  $A = -r_{26}/r_{21}$  and  $B = 0$ . The equality  $B = 0$  is an remarkable property, because from it follows the commutation relation

$$T_R D_c = D_c T_R, \quad (10)$$

which is very important for all our further considerations.

The dispersion decomposition (5) is also applicable to the mirror antisymmetric cell, if one assumes that  $r_{12} \neq 0$  and takes  $A = 0$  and  $B = -r_{16}/r_{12}$ . The commutation relation in this case is the relation  $T_A D_c = D_c T_A$ .

## ABERRATIONS OF PERIODIC SYSTEMS

Let us consider a system constructed by a repetition of  $n$  identical cells and let us assume that the cell matrix  $M_c$  allows dispersion decomposition. Then, using representation (7) for the cell map  $\mathcal{M}_c$ , the map of the repetitive  $n$ -cell system can be expressed as follows

$$: \mathcal{M}_{nc} : = : \mathcal{M}_c :^n$$

$$=_3 \exp(: \mathcal{S}_{3,4}(D_c^{-1} \mathbf{z}) + \mathcal{W}_4(D_c^{-1} \mathbf{z}) :) : M_c^n :. \quad (11)$$

In this representation the aberration functions  $\mathcal{S}_{3,4}$  and  $\mathcal{W}_4$  are given by the following formulas

$$\mathcal{S}_{3,4}(\mathbf{z}) = \sum_{m=0}^{n-1} \mathcal{P}_{3,4}^c(\hat{M}_c^m \hat{\mathbf{z}}, \varepsilon), \quad (12)$$

$$\mathcal{W}_4(\mathbf{z}) = \frac{1}{2} \sum_{l=0}^{n-2} \sum_{j=l}^{n-2} \{ \mathcal{P}_3^c(\hat{M}_c^l \hat{\mathbf{z}}, \varepsilon), \mathcal{P}_3^c(\hat{M}_c^{j+1} \hat{\mathbf{z}}, \varepsilon) \}, \quad (13)$$

where the binary operation  $\{*,*\}$  is the Poisson bracket.

If, additionally, the basic cell is mirror symmetric with the map  $\mathcal{M}$  of its first part given by the Lie factorization

$$: \mathcal{M} : =_3 \exp(: \mathcal{F}_{3,4}(\mathbf{z}) :) : M : , \quad (14)$$

then the functions  $\mathcal{P}_3^c$  and  $\mathcal{P}_4^c$  in the formulas (12) and (13) can be re expressed using the commutation relation (10) through the function  $\mathcal{P}_{3,4}(\mathbf{z}) = \mathcal{F}_{3,4}(D_c \mathbf{z})$  as follows

$$\mathcal{P}_3^c(\mathbf{z}) = \mathcal{P}_3(\hat{\mathbf{z}}, \varepsilon) + \mathcal{P}_3(\hat{T}_R \hat{M}_c \hat{\mathbf{z}}, \varepsilon), \quad (15)$$

$$\begin{aligned} \mathcal{P}_4^c(\mathbf{z}) &= \mathcal{P}_4(\hat{\mathbf{z}}, \varepsilon) + \mathcal{P}_4(\hat{T}_R \hat{M}_c \hat{\mathbf{z}}, \varepsilon), \\ &+ (1/2) \{ \mathcal{P}_3(\hat{\mathbf{z}}, \varepsilon), \mathcal{P}_3(\hat{T}_R \hat{M}_c \hat{\mathbf{z}}, \varepsilon) \}, \end{aligned} \quad (16)$$

where the  $4 \times 4$  matrix  $\hat{T}_R$  is the upper left block of the matrix  $T_R$ .

The formulas (15) and (16) are also applicable to the mirror antisymmetric cell with simple exchange of the matrix  $\hat{T}_R$  with the matrix  $\hat{T}_A$  and thus in the following we will restrict our consideration to the mirror symmetric case only.

## GROUPS AND ACHROMATS

Let us assume that the cell matrix  $M_c$  allows dispersion decomposition (5). Then, according to the formulas (12) and (15), the function  $\mathcal{S}_3$ , which is responsible for the second order aberrations of the repetitive  $n$ -cell system, can be represented in the form

$$\mathcal{S}_3(\hat{\mathbf{z}}, \varepsilon) = n \cdot \mathcal{R}_C(\mathcal{P}_3^c(\hat{\mathbf{z}}, \varepsilon)), \quad (17)$$

$$\mathcal{R}_C(f(\hat{\mathbf{z}}, \varepsilon)) = \frac{1}{n} \sum_{m=0}^{n-1} f(\hat{M}_c^m \hat{\mathbf{z}}, \varepsilon) \quad (18)$$

for the basic cell without symmetries and in the form

$$\mathcal{S}_3(\hat{\mathbf{z}}, \varepsilon) = 2n \cdot \mathcal{R}_D(\mathcal{P}_3(\hat{\mathbf{z}}, \varepsilon)), \quad (19)$$

$$\mathcal{R}_D(f) = \frac{1}{2n} \sum_{j=0}^{n-1} \left( f(\hat{M}_c^j \hat{\mathbf{z}}, \varepsilon) + f(\hat{T}_R \hat{M}_c^{j+1} \hat{\mathbf{z}}, \varepsilon) \right) \quad (20)$$

for the mirror symmetric basic cell.

### Appearance of the Cyclic Group $C_n$

For the general midplane symmetric cell the polynomial  $\mathcal{P}_3^c$  can have as much as 18 nonzero monomials responsible for the independent second order transverse aberrations. Why should one expect that the polynomial  $\mathcal{S}_3$  defined by the equality (17) has a smaller number of them, i.e. why should one expect that the map of the  $n$ -cell system has less independent second order aberrations than the cell map? No reason is seen for that in the case of an arbitrary matrix  $\hat{M}_c$ . The situation will change if the matrices

$$I, \hat{M}_c, \dots, \hat{M}_c^{n-1} \quad (21)$$

will form a finite matrix group. Then the operator  $\mathcal{R}_C$  becomes the Reynolds (averaging) operator of this group and, therefore, the polynomial  $\mathcal{S}_3$  is not an arbitrary polynomial anymore. It becomes (as a result of averaging) a polynomial which is invariant under the group action. Moreover, the functions  $\mathcal{S}_4$  and  $\mathcal{W}_4$ , which are responsible for the third order aberrations, will satisfy the identities

$$\mathcal{S}_4(\hat{\mathbf{z}}, \varepsilon) = \mathcal{R}_C(\mathcal{S}_4(\hat{\mathbf{z}}, \varepsilon)), \quad (22)$$

$$\begin{aligned} \mathcal{W}_4(\hat{\mathbf{z}}, \varepsilon) &= \mathcal{R}_C(\mathcal{W}_4(\hat{\mathbf{z}}, \varepsilon)) \\ &+ \sum_{m=0}^{n-2} (n-m-1) \{ \mathcal{P}_3^c(\hat{M}_c^m \hat{\mathbf{z}}, \varepsilon), \mathcal{R}_C(\mathcal{P}_3^c(\hat{\mathbf{z}}, \varepsilon)) \}, \end{aligned} \quad (23)$$

from which it follows that if the function  $\mathcal{S}_3$  does not depend on the variables  $\hat{\mathbf{z}}$ , then the functions  $\mathcal{S}_4$  and  $\mathcal{W}_4$  are also invariants under the group action.

It is clear that the only group which the matrices (21) can form is a cyclic group, and, in order to have a cyclic group of order  $n$  and not several copies of some cyclic group of smaller dimension, we assume in the following that

$$\hat{M}_c^n = I \quad \text{and} \quad \hat{M}_c^m \neq I \quad \text{for} \quad m = 1, \dots, n-1. \quad (24)$$

Besides that we assume that the  $2 \times 2$  horizontal focusing block of the matrix  $\hat{M}_c$  is not equal to the identity matrix, which, when combined with the equality  $\hat{M}_c^n = I$ , gives sufficient conditions for the existence of the periodic cell dispersion and for the property of the overall transfer matrix of the  $n$ -cell system to be a linear achromat.

### Appearance of the Dihedral Group $D_n$

If the basic cell is mirror symmetric, then good cancellation of aberrations can be achieved if the group will be formed by the set of matrices

$$I, \hat{M}_c, \dots, \hat{M}_c^{n-1}, \hat{T}_R \hat{M}_c, \dots, \hat{T}_R \hat{M}_c^n. \quad (25)$$

What kind of group could it be? One can check that while the last  $n$  matrices in (25) are antisymplectic, the first  $n$  matrices coincide with the set (21) and are symplectic, and thus must form a subgroup. So let us assume again that the conditions (24) are satisfied and the subset of the first  $n$  matrices of the set (25) is isomorphic to the cyclic group  $C_n$ . Then the set (25) becomes isomorphic to the dihedral group  $D_n$  consisting of  $n$  reflections,  $n-1$  rotations and the identity transformation, and the analogies of the formulas (22) and (23) take on the forms

$$\mathcal{S}_4(\hat{z}, \varepsilon) = \mathcal{R}_D(\mathcal{S}_4(\hat{z}, \varepsilon)), \quad (26)$$

$$\mathcal{W}_4(\hat{z}, \varepsilon) = \mathcal{R}_D(\mathcal{W}_4(\hat{z}, \varepsilon)) + 2 \sum_{j=0}^{n-2} (n-j-1) \{ \mathcal{P}_3^c(\hat{M}_c^j \hat{z}, \varepsilon), \mathcal{R}_D(\mathcal{P}_3(\hat{z}, \varepsilon)) \}. \quad (27)$$

### Optimal Choice of the Cell Phase Advances

From the equalities (17), (22) and (23) it follows that the number of constraints which one has to satisfy (typically, with usage of multipoles) in order to make repetitive  $n$ -cell system a second or third order achromat is defined by the number of distinct invariant homogeneous polynomials in the variables  $\hat{z}$  of the cyclic group  $C_n$  generated by the matrix  $\hat{M}_c$ . As an abstract object the group  $C_n$  is unique and for all possible matrices  $\hat{M}_c$  satisfying (24) we have groups which are isomorphic to each other, but not all of them are conjugate. Thus as groups of symmetries they can be distinct and can have different number of uncanceled second ( $a_2$ ) and third ( $a_3$ ) order aberrations, and this depends on the choice of the periodic cell phase advances  $\mu_{x,y}^c$ . Due to our assumptions about the matrix  $\hat{M}_c$

$$\mu_{x,y}^c = 2\pi q_{x,y} / n \pmod{2\pi} \quad (28)$$

for some  $q_x = 1, \dots, n-1$  and some  $q_y = 0, \dots, n-1$  such that the numbers  $n, q_x, q_y$  have no common positive factor other than 1.

For arbitrary  $\mu_{x,y}^c$  the numbers  $a_2$  and  $a_3$  cannot be smaller than two and five respectively (chromatic and non-linear contributions to the cell tunes which cannot be canceled only by group averaging), and their actual values depend on the number of resonances which  $\mu_{x,y}^c$  satisfy and

Table 1: The pairs  $(q_x, q_y)$  which minimize the sum  $a_2 + a_3$  of the number of independent transverse second and third order aberrations of the midplane symmetric  $n$ -cell system.

$n$	$a_2$	$a_3$	$(q_x, q_y)$
2	6	25	(1, 0), (1, 1)
3	6	11	(1, 1), (1, 2)
4	2	13	(1, 1), (1, 3)
5	2	7	(1, 1), (1, 4)
6	2	7	(1, 1), (1, 2), (1, 4), (1, 5)
7	2	5	(1, 2), (1, 5)
8	2	7	(1, 1), (1, 2), (1, 3), (1, 5), (1, 6), (1, 7)
9	2	5	(1, 2), (1, 3), (1, 6), (1, 7)
10	2	5	(1, 2), (1, 3), (1, 7), (1, 8)

which are not forbidden by the midplane symmetry. The phase advances which are optimal for the construction of the repetitive third order achromats are given in the table 1 for  $n = 2, \dots, 10$ . Note that in the last column of this table only the generating pairs  $(q_x, q_y)$  are shown. In order to obtain all optimal pairs one has to multiply generating pairs by all positive integers which are smaller than  $n$  and are coprime to  $n$  using modulo  $n$  arithmetic. For example, for  $n = 8$  we have in this table the pair  $(q_x, q_y) = (1, 3)$ . This pair, when multiplied by 3, 5 and 7 using modulo 8 arithmetic, generates three more optimal pairs (3, 1), (5, 7) and (7, 5).

If the basic cell is mirror symmetric, then, for the same cell tunes, the numbers of remaining transverse second ( $\hat{a}_2$ ) and third ( $\hat{a}_3$ ) order aberrations are given by the relations

$$\hat{a}_2 = 2 + (a_2 - 2) / 2, \quad \hat{a}_3 = 5 + (a_3 - 5) / 2. \quad (29)$$

One sees that if  $a_2 > 2$  or  $a_3 > 5$ , then the usage of the mirror symmetry reduces the number of aberrations left for correction by multipoles. So if the number of the independent multipole families required for the aberration correction is of concern, then it could be better to use achromats based on the  $D_n$  group. But if the total number of multipole magnets has to be minimized, then achromats utilizing the  $C_n$  group perform better, because in the mirror symmetric case one has to put multipoles into the basic cell in such a way that the symmetry is preserved.

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